On Differential Galvanometers,

by Louis Schwendler, *Esq.*

There is one very interesting question connected with the construction of these instruments which, as far as I know, has not yet been answered, and which is of sufficient practical importance to form the subject of an investigation.

This question may best be put as follows :

*A certain battery of given electromotive force and given internal resistance has to supply the two coils of any differential galvanometer with a current ; what must be the resistance of either coil in order to obtain the most delicate reading when measuring a given resistance ?**

The solution of this problem in its most general form would naturally be extremely intricate, and could not be effected without tedious calculation, but there is one special case where it is comparatively easy to determine the law which connects the resistance of the coils with the external resistances to be compared, in order to have the greatest sensitiveness of the instrument.

Suppose for instance that the two coils of a differential galvanometer have equal resistances and equal magnetic momenta, and further that the battery which supplies the two coils with current has an internal resistance sufficiently small to allow of its being neglected against the resistances to be compared. Then, on account of the battery resistance being so small, it follows that the current through one coil is entirely independent of the total resistance in the other, and as the two coils are supposed to have equal magnetic momenta and equal resistances, balance can only be established by the currents becoming equal, that is to say at or near balance each coil must receive a current

$$C = \frac{E}{g + w}$$

where g is the unknown resistance of either coil,

w the resistance to be measured, and which is supposed to be known,

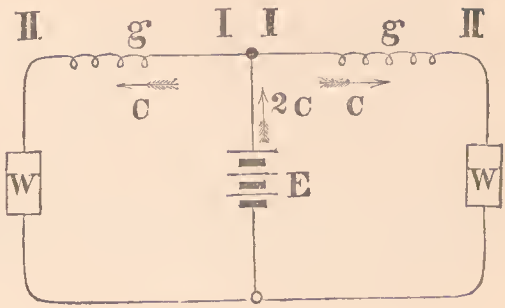
and E the given electromotive force of the testing battery.

At balance the diagram of this differential galvanometer is, therefore, represented by Fig. 1.

* In the Philosophical Magazine of May, 1866, and January, 1867, I solved a similar question, *viz.* the proper resistance of the galvanometer to be employed when testing by Wheatstone's balance, and the result of that investigation has led me to examine the present question.

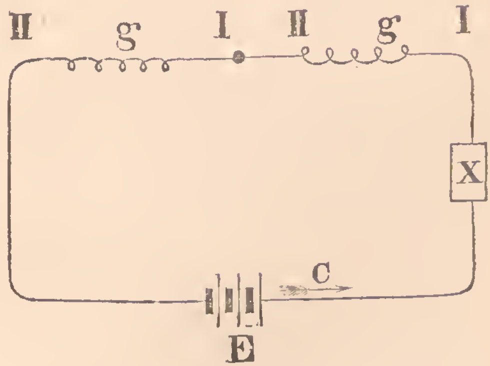
Now, as far as the magnetic effect of the two coils is concerned, we may substitute for the parallel circuit, Fig. 1, the simple circuit, Fig. 2, if we only reverse the magnetic action of one of the two coils, (say the right one).

Fig. 1.



And in order to have, in this case, for the same electromotive force E the same current C flowing through the coils as before, (see Fig. 1), we must necessarily introduce a resistance x hence—

(Fig. 2).



$$C = \frac{E}{g + w} = \frac{E}{2g + x}$$

therefore $w = g + x$ .... (I)

But to obtain the maximum magnetic effect in any single circuit (Fig. 2), it is necessary that the resistance of the coil should be equal to the total external resistance* and therefore in this case (Fig. 2)

$x = 2g$ .... (II)

Eliminating x from equation I and II we have

$g = \frac{w}{3}$ .... (I)

To obtain the most delicate reading with a differential galvanometer, the two coils of which have equal magnetic momenta, and also equal resistances,

* This law holds good,—as can easily be shown,—for any number of coils connected into a single circuit, no matter if the magnetic effects of these coils have the same or opposite sign with respect to a given magnetic point.

the resistance of each coil should always be the third part of the resistance to be measured.

This relation is so exceedingly simple that at first I thought it must be a well known one, and that I only was unacquainted with it. However, I have since carefully read the literature on the subject, and I find the above law nowhere stated, and as a further proof of its being new, I may add that none of the differential galvanometers with which I have had occasion to deal, fulfil it. That this relation is of the greatest importance in the construction of differential galvanometers cannot be doubted, and I have accordingly thought it worth while to bring my investigation before the Society.

Solution of the Problem in its most general form.

Fig. 3.

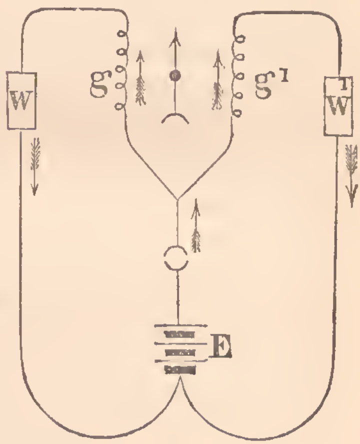


Fig. 3 gives the diagram of a differential galvanometer in its general form. w and w' are the two resistances to be compared and which we suppose to be given. E is the given electromotive force of the testing battery, and f the total resistance in the battery branch ;— g and g' are the resistances of the two coils, and their values are to be determined under the condition that the reading, when near balance, is most delicate, *i. e.* that the slightest variation in w or w' causes the greatest possible variation in the deflection of the needle.

The magnetic moment of the coil g, when a current G passes through it, may be designated by Y, and the magnetic moment of the coil g', when a current G$'$ passes through it, may be called Y$'$. Both these magnetic momenta are taken with respect to the same needle, or system of needles, and we may suppose that neither Y nor Y$'$ alter perceptibly, when the needle, or system of needles, slightly alters its position towards the coils, which are supposed to be fixed. (This condition will be fulfilled as closely as possible near balance, when the needle is approximately always in the same position with respect to the coils, and it is only for such a case that the following investigation is of any practical interest).

According to the principle of the differential galvanometer, we have—

$$a^\circ \propto Y - Y'$$

where a represents the deflection of the needle, before balance is arrived at, and which may be positive, zero or negative, depending on the relative strength of the currents which at the time are acting through the coils, on the relative position of the needle towards the coils, and on the shape and size of the latter.

Approximately we have further

$$Y = m \ U \ G$$
$$Y' = m' \ U' \ G'$$

U and U$'$ being the number of convolutions in the coils g and g' respectively, and m, m' representing the magnetic momenta of an average convolution (one of mean size and mean distance from the needle) in the coils g and g' respectively, when a current of unit strength passes through them.

Further, as the space of each coil to be filled with wire of constant conductivity is given, we have—

$$U = n \sqrt{g}$$
$$U' = n' \sqrt{g'}$$

as can be easily proved.

n and n' are quantities independent of g and g', so long as it may be allowed to neglect the thickness of the insulating covering of the wire against its diameter, which for brevity's sake we will suppose to be the case. With this reservation n and n' depend entirely on the size of the coils and on the manner of coiling.

Substituting these values, we get

$$a^\circ \propto m \ n \sqrt{g} \ G - m' \ n' \sqrt{g'} \ G' \ \ldots\ldots\ldots\ldots \quad I$$

which general expression for the deflection we may write in two different forms either

$$a^\circ \propto m \ n \sqrt{g} \left(G - \frac{m' \ n'}{m \ n} \ \frac{\sqrt{g'}}{\sqrt{g}} \ G' \right) \ldots\ldots \quad I$$

or

$$a^\circ \propto m' \ n' \sqrt{g'} \left(\frac{m \ n}{m' \ n'} \ \frac{\sqrt{g}}{\sqrt{g'}} \ G - G' \right) \ \ldots \quad I'$$

which means that any deflections observed may be naturally considered due to either coil. In the first form (equation I) it is considered due to the coil g, when a current $G - \frac{m' \ n'}{m \ n} \ \frac{\sqrt{g'}}{\sqrt{g}} \ G'$ flows through it, in the latter form (equation I$'$) it is considered due to the coil g', when a current $\frac{m \ n}{m' \ n'} \ \frac{\sqrt{g}}{\sqrt{g'}} \ G - G'$ flows through it.

Now considering that the same battery E has to supply the current to both the coils we have

$$G = E \frac{g' + w'}{N}$$

$$\text{and } G' = E\,\frac{g+w}{N}$$

$$\text{where } N = (g+w)(g'+w') + f(g+w+g'+w').$$

Thus substituting in I and I′ we get either

$$a^\circ \propto m\,n\,E\,\frac{\sqrt{g}}{N}\overbrace{\left(g'+w' - \frac{m'\,n'}{m\;\,n}\,\frac{\sqrt{g'}}{\sqrt{g}}\,(g+w)\right)}^{\Delta} \quad\ldots\ldots\ldots\ldots\quad \text{I}$$

$$\text{or } a^\circ \propto m'\,n'\,E\,\frac{\sqrt{g'}}{N}\overbrace{\left((g'+w')\,\frac{m\;\,n}{m'\,n'}\,\frac{\sqrt{g}}{\sqrt{g'}} - (g+w)\right)}^{\Delta'} \quad\ldots\ldots\ldots\quad \text{I}'$$

and either Δ or Δ' is the factor which at balance becomes zero.

The coefficient $\dfrac{m'\,n'}{m\;\,n}\,\dfrac{\sqrt{g'}}{\sqrt{g}}$ means, therefore, nothing else than what is generally called the constant of the differential galvanometer, *i. e.*, the number by which the total resistance in one branch of the differential galvanometer has to be multiplied, in order to obtain the total resistance in the other branch, when balance is established. This constant of the differential galvanometer is a given function of g and g', the resistance of the coils, and as g and g' are to be determined, by being variable, it cannot be considered a constant in this investigation. But the factor $\dfrac{m'\,n'}{m\;\,n}$ is entirely independent of any of the resistances, it represents what may appropriately be called the '*mechanical arrangement*' of the differential galvanometer, and may be designated by p. It must be borne in mind that p represents an absolute number, which theoretically may be anything with the exception of o and ∞. If p has a value equal to either of these two limits, the instrument would be a simple galvanometer with a shunt, and not a differential galvanometer.

The deflection a may now be written more simply, as follows:—

$$a^\circ \propto K\,\frac{\sqrt{g}}{N}\overbrace{\left(g'+w' - p\,\frac{\sqrt{g'}}{\sqrt{g}}\,(g+w)\right)}^{\Delta} = K\,\frac{\sqrt{g}}{N}\,\Delta \quad\ldots\ldots\ldots\quad \text{I}$$

$$\text{or } a^\circ \propto K'\,\frac{\sqrt{g'}}{N}\overbrace{\left(\frac{g'+w'}{p}\,\frac{\sqrt{g}}{\sqrt{g'}} - (g+w)\right)}^{\Delta'} = K'\,\frac{\sqrt{g'}}{N}\,\Delta' \quad\ldots\ldots\ldots\quad \text{I}'$$

K and K′ being independent of g and g', and also of w and w'.

N is a known function of all the resistances in the differential circuit.

Δ and Δ' are similar functions of g and g', w and w' and which functions become both zero at balance.

For the further investigation, only one of the two possible expressions of a will be used, *viz.* equation I.

$$a^\circ \propto K \frac{\sqrt{g}}{N} \Delta \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad I$$

Differentiating this expression with respect to w', the external resistance belonging to the coil g', we get

$$\frac{da}{dw'} = K \left\{ \frac{\sqrt{g}}{N} - \frac{\Delta R \sqrt{g}}{N^2} \right\}$$

$$\text{where } R = \frac{dN}{dw'}$$

or the variation of the deflection a, when w' varies, is

$$\delta a = K \left\{ \frac{\sqrt{g}}{N} - \frac{\Delta R \sqrt{g}}{N^2} \right\} dw' = K \phi\, dw'.$$

Now it is clear that the instrument is most sensitively constructed when, for the slightest variation in w', the variation in a is greatest. This will be the case if the factor $\phi = \frac{\sqrt{g}}{N} - \frac{\Delta R \sqrt{g}}{N^2}$ is as great as possible. This factor ϕ is a known function of the resistances in the circuit, and as w and w' are given, ϕ can only be made a maximum with respect to g and g', the resistances of the two coils.

Thus our physical problem is reduced to the following mathematical one :

A function ϕ containing two variables is to be made a maximum, while the two variables are fixed to each other by the relation

$$\Delta = g' + w' - p \frac{\sqrt{g'}}{\sqrt{g}} (g + w),$$

Δ being a constant with respect to g and g' and becoming zero at balance.

Solving this question (relative maxima), we get

$$\frac{(w-g)\,(w'+g') + f(w + w' + g' - g)}{p\,(g - w)\,g'} = \frac{2\,(g + w + f)}{2\sqrt{g}\sqrt{g'} - p\,(g + w)} \ldots \text{II.*}$$

* To some of the readers, a more detailed working out of the mathematical problem may, perhaps, be welcome; and as this will also prove to be an easy control over the equations (II) and (II′), I will give it here in a somewhat condensed form. We had

$$a^\circ \propto K \frac{\sqrt{g}}{N} \Delta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad I$$

where K represents a constant, *i. e.* a quantity independent of any of the resistances in the differential circuit (Fig. 3), while $\Delta = g' + w' - p \frac{\sqrt{g'}}{\sqrt{g}} (g + w)$, *i. e.* a resistance which at balance becomes $= 0$; and further

$N = (g + w)(g' + w') + f\,(g + w + g' + w')$.

Differentiating a with respect to w', and remembering that $\frac{d\Delta}{dw'} = 1$, and substituting $\frac{dN}{dw'} = R$, we have which equation with the other

$$g' + w' - p \frac{\sqrt{g'}}{\sqrt{g}} (g + w) = \Delta = 0 \ \ldots\ldots\ldots\ldots\ 1$$

gives all that is required to determine g and g', and the values thus obtained

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$$\frac{d\,a}{d\,w'} = \mathrm{K} \left\{ \frac{\sqrt{g}}{\mathrm{N}} - \Delta \frac{\mathrm{R}\sqrt{g}}{\mathrm{N}^2} \right\}$$

$$\therefore \quad \delta a = \mathrm{K} \underbrace{\left\{ \frac{\sqrt{g}}{\mathrm{N}} - \Delta \frac{\mathrm{R}\sqrt{g}}{\mathrm{N}^2} \right\}}_{\phi} \delta w'$$

$$\therefore \quad \delta a = \mathrm{K}\, \phi\, \delta w'$$

Thus the variation of a is always directly proportional to ϕ, a known function of g and g', and to make δa for any $\delta w'$ as large as possible, we have to make ϕ a maximum with respect to g and g', whilo g and g' are connected by the following equation

$$\Delta = g' + w' - p \frac{\sqrt{g'}}{\sqrt{g}} (g + w) \ \ldots\ldots\ldots\ldots\ \mathrm{I}$$

p being a constant with respect to g and g', as also is Δ.

We have, therefore, to deal here with a relative maximnm, and in accordance with well known rules, we have to form the following partial differential coefficients:

$$\frac{d\,\phi}{d\,g} = \left\{ \frac{\mathrm{N} - 2g \frac{d\,\mathrm{N}}{d\,g}}{2\sqrt{g}\,\mathrm{N}^2} - \frac{\mathrm{R}\sqrt{g} \frac{d\,\Delta}{d\,g}}{\mathrm{N}^2} + \Delta\,\mathrm{S} \right\}$$

$$\mathrm{R} = \frac{d\,\mathrm{N}}{d\,w'} = g + w + f$$

$$\mathrm{S} = \frac{\sqrt{g}}{\mathrm{N}^2} \left\{ \frac{2\,\mathrm{R} \frac{d\,\mathrm{N}}{d\,g}}{\mathrm{N}} - \frac{d\,\mathrm{R}}{dg} - \frac{\mathrm{R}}{2\,g} \right\}$$

$$\frac{d\,\phi}{d\,g'} = - \left\{ \frac{\sqrt{g} \frac{d\,\mathrm{N}}{d\,g'}}{\mathrm{N}^2} + \frac{\mathrm{R}\sqrt{g} \frac{d\,\Delta}{d\,g'}}{\mathrm{N}^2} + \Delta\,\mathrm{S}' \right\}$$

$$\mathrm{S}' = \frac{\sqrt{g}}{\mathrm{N}^2} \left(\frac{d\,\mathrm{R}}{d\,g'} - \frac{2\,\mathrm{R} \frac{d\,\mathrm{N}}{d\,g'}}{\mathrm{N}} \right)$$

$$\frac{d\,\Delta}{d\,g} = \frac{w - g}{g} \, \frac{p}{2} \, \frac{\sqrt{g'}}{\sqrt{g}}$$

$$\frac{d\,\Delta}{d\,g'} = \frac{2\sqrt{g}\sqrt{g'} - p\,(g + w)}{2\sqrt{g}\sqrt{g'}}.$$

At or noar balanco when Δ is $= 0$, or very small, the terms $\Delta\,\mathrm{S}$ and $\Delta\,\mathrm{S}'$ in the respective differential coefficients are to bo neglected, because neither S nor S′ become infinito for any finite values of g and g'.

Thus wo havo approximatoly:

$$\frac{d\,\phi}{d\,g} = \frac{\mathrm{N} - 2g \frac{d\,\mathrm{N}}{d\,g}}{2\sqrt{g}\,\mathrm{N}^2} - \frac{\mathrm{R}\sqrt{g} \frac{d\,\Delta}{d\,g}}{\mathrm{N}^2} = \mathrm{P} - \mathrm{Q}$$

would be those which would make the reading most delicate near balance, when the variation takes place in w', *i. e.*, the external resistance belonging to the coil g'.

If, instead of differentiating the expression for a with respect to w' by using the expression I, we had done so with respect to w by using the expression I′, we should have obtained in a similar way the following relation between g and g'

$$\frac{(w'-g')(w+g)+f(w+w'+g-g')}{\frac{g}{p}(g'-w')}=\frac{2(g'+w'+f)}{2\sqrt{g}\sqrt{g'}-\frac{g'+w'}{p}}, \quad \ldots\ldots \text{II}'$$

which equation connected with the other

$$\frac{d\phi}{dg'}=-\left\{\frac{\sqrt{g}\frac{dN}{dg'}}{N^2}+\frac{R\sqrt{g}\frac{d\Delta}{dg'}}{N^2}\right\}=-(P'+Q')$$

further we will substitute:

$$\frac{d\,\Delta}{d\,g}=\alpha$$

$$\frac{d\,\Delta}{d\,g'}=\beta$$

Thus we have the following differential equation:

$$(P-Q)\,dg-(P'+Q')\,dg'+\lambda\left\{\alpha\;dg+\beta\,dg'\right)=0$$

λ being the undetermined factor. From this equation we have:

$$P-Q+\lambda\,\alpha=0$$

and $-(P'+Q')+\lambda\beta=0$

or λ eliminated:

$$-\frac{P-Q}{\alpha}=\frac{P'+Q'}{\beta}$$

but we have always:

$$\frac{Q}{\alpha}=\frac{Q'}{\beta}$$

thus we have as end-equation:

$$-\frac{P}{\alpha}=\frac{P'}{\beta}$$

or the value for P, P′, α and β substituted we have:

$$\frac{N-\frac{2g\,dN}{dg}}{p\,g'(g-w)}=\frac{\frac{2\,d\,N}{d\,g'}}{2\sqrt{g}\sqrt{g'}-p\,(g+w)}$$

further substituting

$$\frac{dN}{dg}=g'+w'+f$$

$$\frac{dN}{dg'}=g\;+w\;+f$$

and reducing as much as possible, we have

$$\frac{(w-g)(w'+g')+f(w+w'+g'-g)}{p\,(g-w)\,g'}=\frac{2\,(g+w+f)}{2\sqrt{g}\sqrt{g'}-p\,(g+w)} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{II}$$

which is the equation II as given above.

In quite a similar manner, equation II′ cau bo found, it must only be remembered that it is more simple to use expression I′ for the purpose than I.

$$\frac{g' + w'}{p} \frac{\sqrt{g}}{\sqrt{g'}} - (g + w) = \Delta' = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad 1$$

gives all that is necessary to determine g and g', being those values which would make the reading at or near balance most sensitive when a variation in w, the external resistance belonging to coil g, takes place.

Now it is clear that equations II and II$'$ are not necessarily identical, as long as p does not fulfil certain conditions, and therefore the first set of equation II and 1 may give entirely different values for g and g' from those obtained from the second set II$'$ and I), which means that a simultaneous maximum sensitiveness with respect to an alteration of the external resistances w, w' in either of the two differential branches, is *not* always possible. The following very important and interesting question, therefore, remains to be solved.

What general condition must be fulfilled in the construction of any differential galvanometer in order to make a simultaneous maximum sensitiveness possible, with respect to an alteration of external resistance in either of the differential branches?

[*To be continued.*]

Notes on a collection of birds from Sikkim,

by W. T. Blanford, F. G. S.—C. M. Z. S.

(With Plates VII and VIII.)

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Mr. L. Mandelli of Darjiling has sent to me for determination a most interesting collection of Sikkim birds, together with a few obtained from the plains near the base of the Himalayas. The birds sent are from various elevations, some being evidently from considerable altitudes. Strange as it may appear, after this chosen land of the feathered tribes had been explored and ransacked for years by such ornithologists as Hodgson, Jerdon, Tickell and many others, it yet yields novelties to so energetic a collector as Mr. Mandelli. Amongst the birds sent is a sixth Himalayan species of *Propasser*, indicated, it is true, some years since by Mr. Blyth, but not hitherto described, and the male of which was previously unknown. There is also a new *Pellorneum*, and apparently one or two undescribed warblers. Two other birds are additions to the fauna of India, and new localities are furnished for a few others.

To my notes on Mr. Mandelli's collection I have added some on birds collected by myself at low elevations in Sikkim. In another paper (antea p. 30), I have given a complete list of all the birds observed or collected by me in the